THE ELASTIC EQUILIBRIUM OF A HYPERBOLOID OF REVOLUTION OF ONE SHEET WITH PRESCRIBED DISPLACEMENTS AT THE BOUNDARY

PMM Vol. 35, No. 4, 1971, pp. 729-734
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(Received January 22, 1970)

The solution of the second fundamental problem of the theory of elasticity is obtained for a hyperboloid of revolution of one sheet. As an example we solve the problem of elastic deformation under the action of a concentrated axial force situated at the center of symmetry of the hyperboloid, under the assumption that the boundary surface is rigidly fixed.

It is proved in [1] that by using oblate spheroidal coordinates and the generalized Mehler-Fock integral expansion, one can obtain the solution of the fundamental problems of the mathematical theory of elasticity for domains bounded by a hyperboloid of revolution of two sheets. In the present paper similar results are obtained for the case of a hyperboloid of revolution of one sheet by using integral expansions with respect to spherical functions which have been considered in [2, 3]. The characteristic property of these expansions is the presence of a discrete part in the spectrum of the eigenvalues and therefore in the expansion of an arbitrary function there exists a finite algebraic sum together with the integral.

1. We consider particular solutions of the equations of the theory of elasticity [1]

$$\frac{1}{1-2\mu}\,\mathrm{grad\,div}\,\mathbf{u}+\Delta\mathbf{u}=0,\qquad \mathbf{u}=\mathbf{i}u+\mathbf{j}v+\mathbf{k}w \tag{1.1}$$

Here $\mathbf{u}$ is the displacement vector and μ is Poisson's ratio.

The first two solutions obtained from the equations

$$\Delta\mathbf{u}=0,\qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1.2}$$

The third solution is constructed with the help of the vector potential $\mathbf{B}$

$$\mathbf{u}=\frac{1}{2G}\left[4(1-\mu)\mathbf{B}-\mathrm{grad}\,(\mathbf{r}\cdot\mathbf{B})\right] \tag{1.3}$$

$$\mathbf{B}=B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k},\qquad \Delta\mathbf{B}=0$$

Here G is the modulus of elasticity.

To solve Eqs. (1.2), (1.3), we make use of the oblate spheroidal coordinates, which are defined by the equations [4]

$$x=c\,\mathrm{ch}\,\alpha\sin\beta\cos\varphi,\quad y=c\,\mathrm{ch}\,\alpha\sin\beta\sin\varphi,\quad z=c\,\mathrm{sh}\,\alpha\cos\beta \tag{1.4}$$

$$(-\infty<\alpha<+\infty,\quad 0<\beta<\beta_0,\quad -\pi<\varphi\leqslant+\pi)$$

The totality of particular solutions of Laplace's equation which are appropriate for the examination of boundary value problems where the boundary conditions are given on the surface of a hyperboloid of one sheet, is of the form [5]

$$u = u_{\nu m} = \begin{matrix}\varphi_\nu^{\ m} (\operatorname{sh} \alpha) \\ \psi_\nu^{\ m} (\operatorname{sh} \alpha)\end{matrix} P_\nu^{-m} (\cos \beta) [M_m (\nu) \cos m\varphi + N_m (\nu) \sin m\varphi] \tag{1.5}$$

$$\varphi_\nu^{\ m} (x) = {}^1/_2 [e^{\mp {}^1/_2 i\pi m} P_\nu^{-m} (ix) + e^{\pm {}^1/_2 i\pi m} P_\nu^{-m} (-ix)] \quad (x \gtrless 0)$$

$$\psi_\nu^{\ m} (x) = -{}^1/_2 i [e^{\mp {}^1/_2 i\pi m} P_\nu^{-m} (ix) - e^{\pm {}^1/_2 i\pi m} P_\nu^{-m} (-ix)] \quad (x \gtrless 0)$$

$$(m = 0, 1, 2, 3, \ldots)$$

Here the parameter ν has a continuous and a discrete spectrum, while $\varphi_\nu^{\ m} (x)$ and $\psi_\nu^{\ m}(x)$ are, respectively, the even and odd combination of spherical functions with imaginary arguments [3].

2. As it follows from (1.2), (1.3), Eq.(1.1) reduces to Laplace's equation for each component of the vectors **u** and **B**.

The particular solutions (1.5) of Laplace's equation admit four kinds of solutions, differing by the type of symmetry with respect to the angle φ and the variable α. For the sake of simplicity, we consider only the case of displacements w which are symmetric with respect to the plane $z = 0$ and the plane $\varphi = 0$. In this case, the solution of Eqs.(1.2) can be obtained by the superposition of particular solutions of the form

$$u_{\nu m}^{(1)} = a_m (\nu) \psi_\nu^{m-1} (\operatorname{sh} \alpha) P_\nu^{-m+1} (\cos \beta) \cos (m-1) \varphi \tag{2.1}$$

$$v_{\nu m}^{(1)} = -a_m (\nu) \psi_\nu^{m-1} (\operatorname{sh} \alpha) P_\nu^{-m+1} (\cos \beta) \sin (m-1) \varphi$$

$$w_{\nu m}^{(1)} = a_m (\nu) (\nu + m) (\nu - m + 1) \varphi_\nu^m (\operatorname{sh} \alpha) P_\nu^{-m} (\cos \beta) \cos m\varphi$$

$$(m = 1, 2, 3, \ldots)$$

$$u_{\nu m}^{(2)} = b_m (\nu) (\nu - m) (\nu + m + 1) \psi_\nu^{m+1} (\operatorname{sh} \alpha) P_\nu^{-m-1} (\cos \beta) \cos (m+1) \varphi$$

$$v_{\nu m}^{(2)} = b_m (\nu) (\nu - m) (\nu + m + 1) \psi_\nu^{m+1} (\operatorname{sh} \alpha) P_\nu^{-m-1} (\cos \beta) \sin (m+1) \varphi \tag{2.2}$$

$$w_{\nu m}^{(2)} = b_m (\nu) \varphi_\nu^m (\operatorname{sh} \alpha) P_\nu^{-m} (\cos \beta) \cos m\varphi$$

To construct the solutions (2.1), (2.2) it is necessary to make use of the recursion relations

$$\frac{d\varphi_\nu^{\ m}}{dx} = -\frac{mx}{x^2+1} \varphi_\nu^{\ m} + \frac{1}{\sqrt{x^2+1}} \psi_\nu^{m-1}$$

$$\frac{d\varphi_\nu^{\ m}}{dx} = \frac{mx}{x^2+1} \varphi_\nu^{\ m} - \frac{(\nu - m)(\nu + m + 1)}{\sqrt{x^2+1}} \psi_\nu^{m+1} \tag{2.3}$$

$$\frac{d\psi_\nu^{\ m}}{dx} = -\frac{mx}{x^2+1} \psi_\nu^{\ m} - \frac{1}{\sqrt{x^2+1}} \varphi_\nu^{m-1}$$

$$\frac{d\psi_\nu^{\ m}}{dx} = \frac{mx}{x^2+1} \psi_\nu^{\ m} + \frac{(\nu - m)(\nu + m + 1)}{\sqrt{x^2+1}} \varphi_\nu^{m+1}$$

The components of the vector potential **B** are obtained by the superposition of particular solutions of the form

$$B_{x\nu m} = -c_m (\nu) (\nu - m) (\nu + m + 1) \psi_\nu^{m+1} (\operatorname{sh} \alpha) P_\nu^{-m-1} (\cos \beta) \cos (m+1) \varphi \tag{2.4}$$

$$B_{y\nu m} = -c_m (\nu) (\nu - m) (\nu + m + 1) \psi_\nu^{m+1} (\operatorname{sh} \alpha) P_\nu^{-m-1} (\cos \beta) \sin (m+1) \varphi$$

$$B_{z\nu m} = c_m (\nu) \operatorname{tg}^2 \beta_0 \varphi (\operatorname{sh} \alpha) P_\nu^{-m} (\cos \beta) \cos m\varphi \qquad (m = 0, 1, 2, \ldots)$$

Substituting (2.4) into (1.3) we obtain for the components of the displacement vector at the boundary $\beta = \beta_0$

$$\begin{matrix} u_{\nu m}^{(3)} \\ \vartheta_{\nu m}^{(3)} \end{matrix} = -c_m(\nu)(\nu - m)(\nu + m + 1)\lambda_m(\nu)\psi_\nu^{m+1}(\operatorname{sh}\alpha)\begin{matrix}\cos(m+1)\varphi \\ \sin(m+1)\varphi\end{matrix} \mp$$

$$\mp {}^1/_2 \operatorname{tg} {}^1/_2\beta_0 c_m(\nu)\psi_\nu^{m-1}(\operatorname{sh}\alpha)\begin{matrix}\cos(m-1)\varphi \\ \sin(m-1)\varphi\end{matrix}$$

$$w_{\nu m}^{(3)} = c_m(\nu)\lambda_m'(\nu)\varphi_\nu{}^m(\operatorname{sh}\alpha)\cos m\varphi \tag{2.5}$$

$$\lambda_m(\nu) = (3-4\mu)P_\nu^{-m-1}(\cos\beta_0) + {}^1/_2 \operatorname{tg}\beta_0(\nu + m + 2)(\nu - m - 1)P_\nu^{-m-2}(\cos\beta_0)$$

$$\lambda_m'(\nu) = \operatorname{tg}^2\beta_0(3-4\mu)P_\nu^{-m}(\cos\beta_0) - \operatorname{tg}\beta_0(\nu + m + 1)(\nu - m)P_\nu^{-m-1}(\cos\beta_0)$$

$$(m = 0, 1, 2, \ldots)$$

$$\nu = \nu_\tau = i\tau - {}^1/_2 \qquad (0 < \tau < \infty)$$

$$\nu = \nu_n = m - 2n - 1 \qquad (n = 0, 1, 2, \ldots, n^*), \qquad n^* = [{}^1/_2(m-1)] \qquad (m = 1, 2, 3, \ldots)$$

Thus, the components of the displacement vectors at the boundary $\beta = \beta_0$ can be written in the form [3]

$$\begin{matrix} u_\rho^{(1)} \\ u_\varphi^{(1)} \end{matrix} = \sum_{m=1}^{\infty}\left\{\int_0^\infty a_m(\tau)\,\psi_{i\tau-1/2}^{m-1}(\operatorname{sh}\alpha)P_{i\tau-1/2}^{-m+1}(\cos\beta_0)\,d\tau\right\}\begin{matrix}\cos m\varphi \\ -\sin m\varphi\end{matrix} +$$

$$+ \sum_{m=3}^{\infty}\left\{\sum_{n=0}^{n^*}\alpha_{mn}\psi_{m-2n-1}^{m-1}(\operatorname{sh}\alpha)P_{m-2n-1}^{-m-1}(\cos\beta_0)\right\}\begin{matrix}\cos m\varphi \\ -\sin m\varphi\end{matrix} \tag{2.6}$$

$$w^{(1)} = -\sum_{m=1}^{\infty}\left\{\int_0^\infty a_m(\tau)\left[\tau^2 + \left(m - \frac{1}{2}\right)^2\right]\varphi_{i\tau-1/2}^{m}(\operatorname{sh}\alpha)P_{i\tau-1/2}^{-m}(\cos\beta_0)\,d\tau\right\}\times$$

$$\times \cos m\varphi + \sum_{m=3}^{\infty}\left\{\sum_{n=0}^{n^*}\alpha_{mn}2n(2n+1-2m)\varphi_{m-2n-1}^{m}(\operatorname{sh}\alpha)P_{m-2n-1}^{-m}(\cos\beta_0)\right\}\cos m\varphi$$

$$\begin{matrix} u_\rho^{(2)} \\ u_\varphi^{(2)} \end{matrix} = -\sum_{m=0}^{\infty}\left\{\int_0^\infty b_m(\tau)\left[\tau^2 + \left(m + \frac{1}{2}\right)^2\right]\psi_{i\tau-1/2}^{m+1}(\operatorname{sh}\alpha)P_{i\tau-1/2}^{-m-1}(\cos\beta_0)\,d\tau\right\}\begin{matrix}\cos m\varphi \\ \sin m\varphi\end{matrix} +$$

$$+ \sum_{m=1}^{\infty}\left\{\sum_{n=0}^{n^*}\beta_{mn}(2n+1)(2n-2m)\psi_{m-2n-1}^{m+1}(\operatorname{sh}\alpha)P_{m-2n-1}^{-m-1}(\cos\beta_0)\right\}\begin{matrix}\cos m\varphi \\ \sin m\varphi\end{matrix} \tag{2.7}$$

$$w^{(2)} = \sum_{m=0}^{\infty}\left\{\int_0^\infty b_m(\tau)\varphi_{i\tau-1/2}^{m}(\operatorname{sh}\alpha)P_{i\tau-1/2}^{-m}(\cos\beta_0)\,d\tau\right\}\cos m\varphi +$$

$$+ \sum_{m=1}^{\infty}\left\{\sum_{n=0}^{n^*}\beta_{mn}\varphi_{m-2n-1}^{m}(\operatorname{sh}\alpha)P_{m-2n-1}^{-m}(\cos\beta_0)\right\}\cos m\varphi \tag{2.8}$$

$$\begin{matrix} u_\rho^{(3)} \\ u_\varphi^{(3)} \end{matrix} = \sum_{m=0}^{\infty}\left\{\int_0^\infty c_m(\tau)\left[\tau^2 + \left(m + \frac{1}{2}\right)^2\right]\lambda_m(\nu_\tau)\psi_{i\tau-1/2}^{m+1}(\operatorname{sh}\alpha)\,d\tau\right\}\begin{matrix}\cos m\varphi \\ \sin m\varphi\end{matrix} +$$

$$+ \sum_{m=1}^{\infty}\left\{\sum_{n=0}^{n^*}\gamma_{mn}(2n+1)(2m-2n)\lambda_m(\nu_n)\psi_{m-2n-1}^{m+1}(\operatorname{sh}\alpha)\right\}\begin{matrix}\cos m\varphi \\ \sin m\varphi\end{matrix} \mp$$

$$\mp\frac{1}{2}\,\mathrm{tg}\,\frac{1}{2}\,\beta_0\sum_{m=0}^{\infty}\left\{\int_0^{\infty}c_m(\tau)\,\psi^{m-1}_{i\tau-1/2}(\mathrm{sh}\,\alpha)\,P^{-m}_{i\tau-1/2}(\cos\beta_0)\,d\tau\right\}\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix}\mp$$

$$\mp\frac{1}{2}\,\mathrm{tg}\,\frac{1}{2}\,\beta_0\sum_{m=3}^{\infty}\left\{\sum_{n=0}^{n^*}\gamma_{mn}\psi^{m-1}_{m-2n-1}(\mathrm{sh}\,\alpha)\,P^{-m}_{m-2n-1}(\cos\beta_0)\right\}\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix}$$

$$w_3=\sum_{m=0}^{\infty}\left\{\int_0^{\infty}c_m(\tau)\,\lambda_m'(\nu_\tau)\,\varphi^{m}_{i\tau-1/2}(\mathrm{sh}\,\alpha)\,d\tau\right\}\cos m\varphi+$$

$$+\sum_{m=1}^{\infty}\left\{\sum_{n=0}^{n^*}\gamma_{mn}\varphi^{m}_{m-2n-1}(\mathrm{sh}\,\alpha)\,\lambda_m'(\nu_n)\right\}\cos m\varphi$$

$$\psi_m{}^m(x)\equiv 0\qquad (m=1,2,3,\ldots)\tag{2.9}$$

3. To solve the second fundamental problem of the theory of elasticity, we will consider taking into account the particular solutions (2.6)-(2.8), that the displacement vector at the boundary $\beta=\beta_0$ is given in the cylindrical system of coordinates ρ, φ, z

$$u_\rho=\sum_{m=0}^{\infty}A_m(\alpha)\cos m\varphi,\qquad u_\varphi=\sum_{m=1}^{\infty}B_m(\alpha)\sin m\varphi$$

$$w=\sum_{m=0}^{\infty}D_m(\alpha)\cos m\varphi\tag{3.1}$$

Here $A_m(\alpha)$ and $B_m(\alpha)$ are odd functions while $D_m(\alpha)$ is an even function of α. We introduce the auxiliary functions

$$f_m^{(+)}(\alpha)=\tfrac{1}{2}[A_m(\alpha)+B_m(\alpha)],\qquad f_m^{(-)}(\alpha)=\tfrac{1}{2}[A_m(\alpha)-B_m(\alpha)]\tag{3.2}$$
$$(m=1,2,3,\ldots)$$

The functions (3.1), (3.2) must satisfy the conditions of the expansion theorem [3]

$$f_m^{(\pm)}(\alpha)=\int_0^{\infty}\bar f_m^{(\pm)}(\tau)\,\psi^{m\pm1}_{i\tau-1/2}(\mathrm{sh}\,\alpha)\,d\tau+\sum_{n=0}^{n^*}\bar f^{(\pm)}_{mn}\psi^{m\pm1}_{m-2n-1}(\mathrm{sh}\,\alpha)\tag{3.3}$$

$$D_m(\alpha)=\int_0^{\infty}\bar D_m(\tau)\,\varphi^{m}_{i\tau-1/2}(\mathrm{sh}\,\alpha)\,d\tau+\sum_{n=0}^{n^*}\bar D_{mn}\varphi^{m}_{m-2n-1}(\mathrm{sh}\,\alpha)$$

Equating (3.1) with the solutions (2.6) - (2.8) at the boundary $\beta=\beta_0$, from (3.2) we obtain for the coefficients $a_m(\tau)$, $b_m(\tau)$, $c_m(\tau)$ the system of algebraic equations

$$a_m(\tau)\,P^{-m+1}_{i\tau-1/2}(\cos\beta_0)-\tfrac{1}{2}c_m(\tau)\,\mathrm{tg}\,\tfrac{1}{2}\beta_0\,P^{-m}_{i\tau-1/2}(\cos\beta_0)=\bar f_m^{(-)}(\tau)\tag{3.4}$$
$$c_m(\tau)\,\lambda_m(\nu_\tau)-b_m(\tau)\,P^{-m-1}_{i\tau-1/2}(\cos\beta_0)=\bar f_m^{(+)}(\tau)\,[\tau^2+(m+\tfrac{1}{2})^2]^{-1}$$
$$a_m(\tau)\,[\tau^2+(m-\tfrac{1}{2})^2]\,P^{-m}_{i\tau-1/2}(\cos\beta_0)+b_m(\tau)\,P^{-m}_{i\tau-1/2}(\cos\beta_0)+c_m(\tau)\,\lambda_m'(\nu_\tau)=\bar D_m(\tau)$$
$$(m=1,2,3,\ldots)$$

To determine the numbers α_{mn}, β_{mn}, γ_{mn} we have the system of algebraic equations

$$\alpha_{mn}P^{-m+1}_{m-2n-1}(\cos\beta_0)-\tfrac{1}{2}\gamma_{mn}\,\mathrm{tg}\,\tfrac{1}{2}\beta_0\,P^{-m}_{m-2n-1}(\cos\beta_0)=\bar f^{(-)}_{mn}\tag{3.5}$$
$$\gamma_{mn}\lambda_m(\nu_n)-\beta_{mn}P^{-m-1}_{m-2n-1}(\cos\beta_0)=(2n+1)^{-1}(2m-2n)^{-1}\bar f^{(+)}_{mn}$$
$$\alpha_{mn}2n\,(2n+1-2m)+\beta_{mn}P^{-m}_{m-2n-1}(\cos\beta_0)+\gamma_{mn}\lambda_m'(\nu_n)=\bar D_{mn}$$
$$(n=1,2,3,\ldots[\tfrac{1}{2}(m-1)];\quad m=3,4,5,\ldots)$$

For the case $n = 0$ the system of algebraic equations can be written in the form

$$\gamma_{m0}\lambda_m(\nu_0) - \beta_{m0}P_{m-1}^{-m-1}(\cos\beta_0) = \bar{f}_{m0}^{(+)} \qquad \begin{pmatrix}\nu_0 = m-1 \\ m = 1, 2, \ldots\end{pmatrix} \tag{3.6}$$
$$\gamma_{m0}\lambda_m'(\nu_0) + \beta_{m0}P_{m-1}^{-m}(\cos\beta_0) = \overline{D}_{m0}$$

4. We consider the case of the axial symmetry of the boundary conditions. In this case $m = 0$ and the expansions (2.6) - (2.8) have only integral terms. In addition, by virtue of (2.3), the solutions (2.1), (2.2) cease to be linearly independent and it is necessary to put $a_0(\tau) = 0$. From the solutions (2.2), (2.5) it is easy to obtain the components of the displacement vector at the boundary $\beta = \beta_0$

$$u_\rho = \int_0^\infty \left(\tau^2 + \frac{1}{4}\right)[c_0(\tau)\lambda_0(\tau) - b_0(\tau)\,P_{i\tau-1/2}^{-1}(\cos\beta_0)]\,\psi_{i\tau-1/2}^1(\operatorname{sh}\alpha)\,d\tau \tag{4.1}$$

$$w = \int_0^\infty [c_0(\tau)\lambda_0'(\tau) + b_0(\tau)\,P_{i\tau-1/2}(\cos\beta_0)]\,\varphi_{i\tau-1/2}(\operatorname{sh}\alpha)\,d\tau$$

Here

$$\lambda_0(\tau) = (3-4\mu)\,P_{i\tau-1/2}^{-1}(\cos\beta_0) - {}^1/_2\operatorname{tg}{}^1/_2\beta_0 \times$$
$$\times\left[P_{i\tau-1/2}(\cos\beta_0) - \left(\tau^2 + \frac{9}{4}\right)P_{i\tau-1/2}^{-2}(\cos\beta_0)\right]$$
$$\lambda_0'(\tau) = \operatorname{tg}^2\beta_0\,(3-4\mu)\,P_{i\tau-1/2}(\cos\beta_0) + \operatorname{tg}\beta_0\,(\tau^2 + {}^1/_4)\,P_{i\tau-1/2}^{-1}(\cos\beta_0)$$

Substituting (4.1) into the boundary conditions (3.1) and making use of the expansion (3.3), we obtain a system of algebraic equations for the determination of the coefficients $b_0(\tau)$, $c_0(\tau)$

$$c_0(\tau)\lambda_0(\tau) - b_0(\tau)\,P_{i\tau-1/2}^{-1}(\cos\beta_0) = (\tau^2 + {}^1/_4)^{-1}\bar{A}_0(\tau) \tag{4.2}$$
$$c_0(\tau)\lambda_0'(\tau) + b_0(\tau)\,P_{i\tau-1/2}(\cos\beta_0) = \overline{D_0}(\tau) \qquad (0 < \tau < \infty)$$

Example. We consider the elastic equilibrium of a hyperboloid of revolution of one sheet under the action of a concentrated axial force P, situated at the center of symmetry and having the boundary $\beta = \beta_0$ rigidly fixed. We divide the components of the displacement vector into two terms

$$u_\rho = u_{\rho 0} - u_{\rho 1}, \qquad w = w_0 - w_1 \tag{4.3}$$

Here u_{ρ_0} and w_0 are displacements created by such a force in the unbounded space [6]

$$u_{\rho 0} = \frac{Q\rho z}{R^3}, \quad w_0 = Q\left(\frac{z^2}{R^3} + \frac{3-4\mu}{R}\right), \quad Q = \frac{P}{16\pi\sigma(1-\mu)}, \quad R = \sqrt{\rho^2 + z^2} \tag{4.4}$$

The displacements u_{ρ_1}, w_1 must satisfy Eq. (1.1) for the boundary conditions $\beta = \beta_0$

$$u_{\rho 1} = A_0(\alpha) = \frac{Q}{c}\,\frac{\operatorname{ch}\alpha\sin\beta_0\operatorname{sh}\alpha\cos\beta_0}{(\operatorname{sh}^2\alpha + \sin^2\beta_0)^{3/2}} \tag{4.5}$$
$$w_1 = D_0(\alpha) = \frac{Q}{c}\left[\frac{\operatorname{sh}^2\alpha\cos^2\beta_0}{(\operatorname{sh}^2\alpha + \sin^2\beta_0)^{3/2}} + \frac{3-4\mu}{(\operatorname{sh}^2\alpha + \sin^2\beta_0)^{1/2}}\right]$$

To find the functions $A_0(\tau)$, $D_0(\tau)$ we make use of the expansion [5]

$$\frac{c}{R} = \frac{1}{(\operatorname{sh}^2\alpha + \sin^2\beta_0)^{1/2}} = \pi\int_0^\infty \frac{\tau\operatorname{th}\pi\tau}{\operatorname{ch}^2\pi\tau}\,P_{i\tau-1/2}(0)\times$$
$$\times[P_{i\tau-1/2}(\cos\beta_0) + P_{i\tau-1/2}(-\cos\beta_0)]\,\varphi_{i\tau-1/2}(\operatorname{sh}\alpha)\,d\tau \tag{4.6}$$

Differentiating (4.6) with respect to the parameters α and β_0 and adding the obtained expansions with the corresponding coefficients, we obtain

$$A_0(\alpha) = \int_0^\infty \overline{A}_0'(\tau)\, \varphi^1_{i\tau-1/2}(\operatorname{sh}\alpha)\, d\tau \tag{4.7}$$

$$A_0'(\tau) = \frac{\pi Q}{c} \sin 2\beta_0 \frac{\tau(\tau^2 + 1/4)\operatorname{th}\pi\tau}{\operatorname{ch}^2\pi\tau} P_{i\tau-1/2}(0)\,[P_{i\tau-1/2}(\cos\beta_0) + P_{i\tau-1/2}(-\cos\beta_0)]$$

$$D_0(\alpha) = \int_0^\infty \overline{D}_0'(\tau)\, \varphi_{i\tau-1/2}(\operatorname{sh}\alpha)\, d\tau \tag{4.8}$$

$$D_0'(\tau) = \frac{2\pi Q}{c} \frac{\tau \operatorname{th}\pi\tau}{\operatorname{ch}^2\pi\tau} P_{i\tau-1/2}(0)\, \{(3 - 4\mu + \cos^2\beta_0) \times$$
$$\times [P_{i\tau-1/2}(\cos\beta_0) + P_{i\tau-1/2}(-\cos\beta_0)] +$$
$$+ 1/2 \sin 2\beta_0 (\tau^2 + 1/4)\,[P^{-1}_{i\tau-1/2}(\cos\beta_0) - P^{-1}_{i\tau-1/2}(-\cos\beta_0)]\}$$

The displacements $u_{\beta 1}$, w_1 at the boundary $\beta = \beta_0$ can be represented in the form of the expansions (4.1). The coefficients $b_0(\tau)$ and $c_0(\tau)$ are determined from the system of equations (4.2), where $\overline{A}_0'(\tau)$ and $\overline{D}_0'(\tau)$ are given by Eqs. (4.7), (4.8).

The author expresses his thanks to N. N. Lebedev and Ia. S. Ufliand for advice during discussions on the paper.

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Translated by E. D.